Variational Junction Conditions in $F(T)$ Gravity

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Abstract

We consider variationally permissible junction conditions in extended teleparallel gravity. The general junction condition requires continuity of the normal component of the boundary term at the junction hypersurface. We show that in the spherically symmetric case, if continuity of the tetrad and spin connection are assumed, both Synge's and ISLD boundary conditions of GR are obtained. We analyze both the static (R-Domain) and Time only (T-Domain) dependent scenarios. With the assumption that $F(T)$ is continuously differentiable, both conditions are obtained in the $F(T) \neq T$ case.

***Keywords* —**

1. Introduction

In 1915 Einstein introduced the scientific community to general relativity. One of the key assumptions in the model was that the theory was torsion free. Since then many other models of gravity have been proposed that use curvature to explain how masses move and affect spacetime. There are models that use both curvature and torsion to explain our observations, the most familiar of these is the Einstein-Cartan theory. Interestingly, it is also possible to construct models that are described in terms of a function of only the torsion. These torsion only theories are called $F(T)$ gravity theories. Rather than masses curving space, masses twist space, this is to say translational defects occur when parallel transporting vectors. This concept is notoriously difficult to visualize in more than one dimension. Imagine a hand crank that moves an object along a spiral path. Once the crank has returned to it's original position, the object is it moving has returned to the same angular position but it is now radially displaced. This is to say that the actual position of the object and the 4 dimensional space it is embedded in are no longer the same thing. In the special case where the model is linear in the torsion scalar, it is equivalent to general relativity except for the boundary term. This is called the Teleparallel equivalent of General Relativity. Both this case and the more complicated case where the Lagrangian consists of a non linear function of the torsion scalar will be examined.

The concept of gravity requires both intrinsic and extrinsic curvature. Intrinsic curvature is a property of a surface that does not change as you deform it without stretching. This is the reason that Mercator projections of the globe fail, they are taking an image on a sphere, and trying to project it onto a flat surface. A sphere has positive

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intrinsic curvature and a flat object has 0 intrinsic curvature. Extrinsic curvature is a property of the specific embedding of a surface. A cylinder has 0 intrinsic curvature, it is indistinguishable from a piece of paper. One interesting implication of this is that an observer on the surface can determine the intrinsic curvature of their space, but not the extrinsic curvature.

Two important boundary conditions arise in general relativity, and it is an interesting question whether or not these conditions can be recovered in $F(T)$ models of gravity. The first is the ISLD condition, which is that continuity of the extrinsic curvature implies continuity of the equations of motion. The second condition, Synge's condition, is that continuity of the normal component of the boundary term implies the continuity of the equations of motion. We show which of these conditions are recovered in the various cases we examine.

2. Theory

Curvature can best be quantified as an angular defect in a vector that has been parallel transported on a surface. The usual example is taking a vector pointing tangent to a sphere and moving it in a circuit, as shown in fig(1). Here we are parallel transporting a vector from point P back to point P.

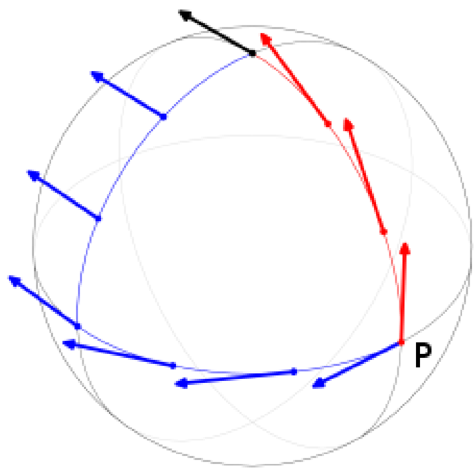


Figure 1: *In a space with curvature but no torsion, we can see that there is a 90° angular difference between the red and blue vectors at point P when parallel transported around a closed loop.*

Torsion is a similar property that manifolds can have. It is best imagined as a twisting around the path of parallel transport. In a surface that has non zero torsion, the path that a vector is transported over matters. Consider a square path on a surface with torsion, moving from one corner to the opposite corner along one path will result in an translational difference compared to moving via the opposite path, see fig(2). As an example moving 4 steps up and 4 steps to the right gets you to the point (3.95,4) and moving 4 steps right and then 4 up gets you to the point (4,3.9)

To understand the objects that build the theory, we need some language from differential geometry. A fibre bundle is a space that appears to be a product space, but

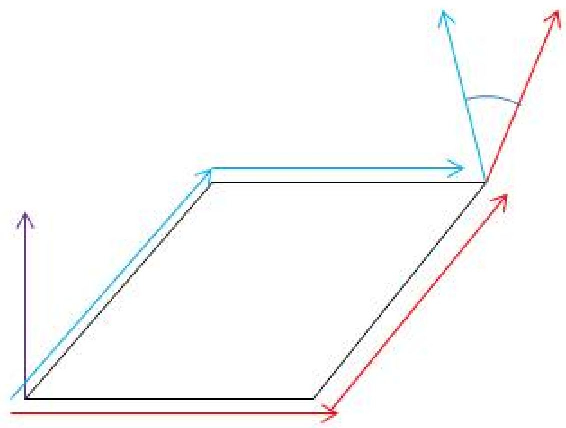


Figure 2: *In a space with torsion but no curvature, we can see that travelling along equivalent paths has produced an angular difference between the red and blue vectors.*

might have some other topology globally. This is similar to a the idea that a space can be locally flat, but have curvature globally. Möbius strips and Klein bottles are familiar examples of fibre bundles.This leads us to the concept of a vector bundle. A vector bundle is an invertible homomorphism that takes a set of vectors from one space to another. A class of these bundles is tangent bundles, which maps a set of vectors to its tangent space. Note that we are assuming that spacetime is a smooth manifold, which is required for the definition of a vector bundle to work.

A connection on a fibre bundle is a device that defines the notion of parallel transport on that bundle.This arises out of the problem of defining rates of change as one vector space approaches another. This is the same concept that is usually called a gauge in physics. In GR the Christoffel connection is used specifically because it is torsion free. This allows the model to be built up using the Ricci tensor. In F(T) theories the Weitzenböck connection for a similar reason: it is curvature free.

Tetrads are a set of four vectors that map the 4D tangent space of each point of a differentiable manifold. The Latin letter indices indicate that the vector set is in the orthonormal tangent space, and the Greek indices specify the point in Lorentz space to which the tangent space is attached. The original F(T) arose from the issue of Lorentz non-covariance. There are accelerating frame effects in gravity that require a special choice of tetrad to avoid. However we can make the theory covariant by adding a correction term. This term is found by turning the gravity off, and seeing what fictitious forces remain. The correction term is then added to remove these non zero terms in the torsion tensor. The benefit of the covariant formulation is that we can use a much larger class of tetrads, including ones that permit time variation. When we operate in the spherically symmetric model we have to choose a very specific tetrad, the diagonal tetrad, to arrive at the relatively simple equations of motion that model produces.

We can now introduce the metric. The metric tensor takes as input two tangent vectors on a surface, and outputs a real valued scalar. With the metric we can define a distance function that measures how far away two points are from each other on our surface. We are specifically using a positive definite, or Reimannian manifold, this results in the useful property that the shortest distance between two points, called a geodesic, is how far an observer would need to travel to go from point a to point b.

The metric can be built from certain tetrads as well. We call these tetrads metric compatible. Later on we will choose specifically these metric compatible tetrads when we are developing the theory. In the spherically symmetric covariant case we can choose our tetrads to just be the square root of the metric. In the non covariant case we need to be a bit more careful with our choice.

The torsion scalar is built out of curvature free Weitzenböck connection. The Weitzenböck connection is built from any non-trivial tetrad as follows

$$\mathring{\Gamma}^{\rho}{}_{\mu\nu} = h_a{}^{\rho}\partial_\nu h^a{}_\mu. \tag{1}$$

Here $\mathring{\Gamma}^{\rho}{}_{\nu\mu}$ is the Weitzenböck connection and $h^a{}_\mu$ is the tetrad. Note that the significance of the $^\circ$ is to differentiate between the Christoffel connection and the Weitzenböck connection. We can then construct the torsion tensor ($T^{\rho}{}_{\mu\nu}$) which by definition is given by the antisymmetric part of the connection.

$$T^{\rho}{}_{\mu\nu} = \mathring{\Gamma}^{\rho}{}_{\nu\mu} - \mathring{\Gamma}^{\rho}{}_{\mu\nu} \tag{2}$$

We can define the contortion tensor ($K^{\mu}{}_{\nu a}$) as follows:

$$K^{\mu}{}_{\nu a} = \frac{1}{2}\left(T^{\nu\mu}{}_a + T_a{}^{\mu\nu} - T^{\mu\nu}{}_a\right). \tag{3}$$

From this we can write the superpotential ($S_a{}^{\mu\nu}$) which when contracted with the torsion tensor gives us the torsion scalar.

$$S_a{}^{\mu\nu} = \frac{1}{2}\left[K^{\mu}{}_{\nu a} + h^{\mu}_a T^{\beta\nu}{}_\beta - h^{\nu}_a T^{\beta\mu}{}_\beta\right]. \tag{4}$$

In the covariant theory there is a slight adjustment to the Torsion tensor. This arises because we need to correct for the fictitious forces produced by accelerating frames.

$$T^a{}_{\gamma\delta} = \tilde{T}^a{}_{\gamma\delta} + \omega^a{}_{b\gamma}h^b_\delta - \omega^a{}_{b\delta}h^b_\gamma. \tag{5}$$

Here ω is the spin connection.

To express the action, we need to define a function of that torsion, $F(T)$. We are going to assume that this function is continuous and differentiable. This implies that' $F'(T)$ exists and is continuous.

The covariant action fully expressed is

$$\begin{aligned} S = \int h\,\delta h^c_p \Big[& h^{-1}\partial_\beta\left(hS^{\alpha\rho}{}_c\right)f'(T) - h^\kappa_c\, T^\zeta{}_{\alpha\kappa}\, S_\zeta{}^{\rho\alpha} f'(T) \\ & -\omega^a{}_{c\sigma}S_a{}^{\rho\sigma} + S^{\alpha\rho}{}_c\,\partial_\beta(T)f''(T) + \frac{1}{4}h^\rho_c f(T)\Big]d^4x \\ & + \int_{\partial B} f'(T)S_c{}^{\alpha\rho}\, h\delta h^c_\rho\, \hat{n}_\alpha\, d^3x. \end{aligned} \tag{6}$$

Note what if the ω terms are zero we have the non-covariant action. In our boundary term, we have a normal vector. This normal is defined as outwardly pointing, and we are going to only consider cases where the field is orientable. This is to say that the outwardly pointing normal is uniquely defined.

3. Non-Covariant Spherically Symmetric F(T) Gravity

In the non-covariant case we are limited to a class of tetrads that have vanishing internal spin connection. Choosing a tetrad is a non-trivial process. All tetrads need to be metric compatible, this is to say that

$$h^a_\mu h_{\nu a} = g_{\mu\nu}. \tag{7}$$

Here the matrix representing

$$g_{\mu\nu} = \begin{bmatrix} A(r)^2 & 0 & 0 & 0 \\ 0 & -B(r)^2 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin(\theta)^2 \end{bmatrix}$$

Now our task is to find a tetrad that is metric compatible but also has vanishing internal spin connection. Note that in the covariant case we don't have to worry about the internal spin connection and we can choose a simpler tetrad. In the non covariant case the matrix representing the tetrad we use is

$$h^a_\mu = \begin{bmatrix} A(r) & 0 & 0 & 0 \\ 0 & B(r)\sin(\theta)\cos(\phi) & r\cos(\theta)\cos(\phi) & -r\sin(\theta)\sin(\phi) \\ 0 & B(r)\sin(\theta)\sin(\phi) & r\cos(\theta)\sin(\phi) & r\sin(\theta)\cos(\phi) \\ 0 & B(r)\cos(\theta) & -r\sin(\theta) & 0 \end{bmatrix}.$$

The spherically symmetric case is arguably the most important special case. By choosing the diagonal tetrad and choosing our second index to be radial, we arrive at our equations of motion. Here $A(r)$ and $B(r)$ are the metric functions.

$$E^1_1 = \frac{(4r(B(r)-2)A_r(r) + A(r)(r^2B(r)^2F(T) + 4B(r) - 4))}{2r^2A(r)B(r)^2} \tag{8}$$

Since in all cases the tetrad is assumed to be continuous, we can note a few facts. Both Synge's condition and the ISLD condition are sufficient to make the boundary term continuous and continuity of the boundary term implies both conditions. Synge's condition is that the continuity of the boundary term implies that the radial pressure is continuous, and in the $F(T) = T$ case that radial pressure is only dependent on the tetrad and the metric functions. The ISLD condition is the same in all cases: continuity of the boundary term implies continuity of the extrinsic curvature.

In the $F(T) \neq T$ case we get a more complicated equation of motion:

$$E^1_1 = \frac{(4r(B(r)-2)A_r(r) + 4A(r)(B(r)-1)F'(T) + F(T)A(r)r^2B(r)^2}{2r^2A(r)B(r)^2} \tag{9}$$

Again we recover both Synge's and the ISLD condition. Noting that the equation of motion can be solved for $F'(T)$, we see that continuity and differentiability of $F(T)$ implies continuity of $F'(T)$.

4. Covariant F(T) Gravity

For the remainder of the thesis we will be working with the covariant model, as it allows us to use a diagonal tetrad, and all results shown for covariant hold for non covariant.

4.1. Timelike

In the time like boundary condition we contract the superpotential in the boundary term with a normal pointing in the radial direction. Because we are in the covariant frame, we can pick the diagonal tetrad:

$$h^a_\nu = \begin{bmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r & 0 \\ 0 & 0 & 0 & -r\sin(\theta) \end{bmatrix}$$

This tetrad is clearly metric compatible. To get vanishing inertial effects, we need to choose a specific omega. The way to find this ω is to set $\alpha(r)$ and $\gamma(r)$ equal to 0. This in effect turns gravity off. We then set each component of the Torsion tensor to be 0 and solve out for the components of ω. We arrive at a very sparse omega:

$$\omega^{12}{}_2 = 1, \omega^{13}{}_3 = \sin(\theta), \omega^{23}{}_3 = \cos(\theta)$$

The spin connection is antisymmetric in its first two indices, leaving us with a total of 6 non zero entries.
With the spin connection calculated, we can turn the gravity back on, and look at the relevant equation of motion.

$$E^1_1 = \frac{(4r(B(r)-2)A_r(r) + 4A(r)(B(r)-1))F'(T) + F(T)r^2(B(r))^2A(r)}{2r^2(B(r))^2A(r)} \tag{10}$$

For our extrinsic curvature we get terms with $A(r)$,$B(r)$ and $A_r(r)$.

$$K^1_1 = 0, K^2_2 = \frac{r}{\sqrt{B(r)}}, K^3_3 = \frac{r\sin^2(\theta)}{\sqrt{B(r)}}, K^4_4 = \frac{A_r(r)}{2\sqrt{B(r)}} \tag{11}$$

We also get these terms in our boundary term, which is the integrand in the last term of equation 6.

$$F^{'}(T)S_a{}^{r\nu}\hat{N}_r \tag{12}$$

$$2,[1,2] = \pm\tfrac{r(A(r)B(r)-A_r(r)r-A(r))}{2A(r)}$$

$$3,[1,3] = \pm\tfrac{r\sin^2(\theta)(A(r)B(r)-A_r(r)r-A(r))}{2A(r)}$$

$$4,[4,1] = \pm\tfrac{A(r)^2(B(r)-1)}{r}$$

Since our boundary is a rank 3 tensor with 64 entries, it is not represented in full. Fortunately it is very sparse. Above are the only non-zero entries. The tensor is also anti symmetric in it's last two indices. The $3[1,3]$ notation indicates that the positive term is in position $3,1,3$ and the negative in $3,3,1$. You'll note that equation 10 is the exact same equation we arrived at in the non covariant formulation. We are able to recover both Synge's condition and the ISLD condition as we were in the non covariant case.

4.2. Spacelike

For spacelike boundaries, we contract the boundary term with a normal pointing in the time direction. Our metric gets adjusted slightly to look like:

$$g_{\mu\nu} = \begin{bmatrix} A(t)^2 & 0 & 0 & 0 \\ 0 & -B(t)^2 & 0 & 0 \\ 0 & 0 & -t^2 & 0 \\ 0 & 0 & 0 & -t^2\sin(\theta)^2 \end{bmatrix}$$

We can again choose the diagonal tetrad:

$$h^a_\nu = \begin{bmatrix} A(t) & 0 & 0 & 0 \\ 0 & -B(t) & 0 & 0 \\ 0 & 0 & -t & 0 \\ 0 & 0 & 0 & -t\sin(\theta) \end{bmatrix}$$

We follow the same procedure for determining omega and we end up with:

$$\omega^{02}{}_2 = 1, \omega^{03}{}_3 = \sin(\theta), \omega^{23}{}_3 = \cos(\theta)$$

The relevant equation of motion in this case is different:

$$E^0_0 = \frac{(4t(A(t)-2)B_t(t) + 4B(t)(A(t)-1))F'(T) + F(T)t^2(A(t))^2B(t)}{2t^2(A(t))^2B(t)} \tag{13}$$

It is clear that this is a similar equation to the timelike boundary, the only difference is that $r \to t$ and $A \leftrightarrow B$. The extrinsic curvature is slightly different, but still contains the terms $A(t)$,$B(t)$ and $B_t(t)$.

$$K^1_1 = \frac{B_t(t)}{2\sqrt{A(t)}}, K^2_2 = \frac{t}{\sqrt{A(t)}}, K^3_3 = \frac{t\sin^2(\theta)}{\sqrt{A(t)}}, K^4_4 = 0. \tag{14}$$

As does the the boundary term:

$$F^{'}(T)S_a{}^{t\nu}\hat{N}_t \tag{15}$$

$$1,[4,1] = \pm\frac{B(t)^2(A(t)-1)}{t}$$

$$2,[4,2] = \pm\frac{t(B(t)A(t)-B_t(t)t-B(t))}{2B(t)}$$

$$3,[4,3] = \pm\frac{t\sin^2(\theta)(B(t)A(t)-B_t(t)t-B(t))}{2B(t)}.$$

We again recover the ISLD condition and Synge's condition.

4.3. Mixed Boundary Conditions

In the case of mixed boundary conditions, we are no longer able to recover either the ISLD condition or Synge's conditions.The equations of motion contain partial derivatives of the metric functions with respect to both r and t that do not show up in either the boundary term or the extrinsic curvature.

5. Conclusion

To summarize, by varying the action with respect to the tetrad, we were able to show that the ISLD condition is recovered in the spherically symmetric case of torsional gravity with both timelike and spacelike boundaries. This is true for actions containing an arbitrary function for Torsion. Furthermore we were able to show that Synge's condition is recovered in the case where the action is linear in torsion, and, under the assumption that $F'(T)$ is continuous, non linear as well. We were not able to recover either condition in mixed boundary conditions.

5.1. R

$$Eom_1^1 = \frac{(4r(B(r)-2)(A_r(r)) + 4A(r)(B(r)-1))(F^{'}(T) + F(T)r^2B(r)^2A(r)}{2r^2B(r)^2A(r)} \tag{16}$$

Extrinsic Curvature

$$K_1^1 = 0, K_2^2 = \frac{r}{\sqrt{B(r)}}, K_3^3 = \frac{r\sin^2(\theta)}{\sqrt{B(r)}}, K_4^4 = \frac{A_r(r)}{2\sqrt{B(r)}} \tag{17}$$

Boundary Term

$$F'(T){S_a}^{r\nu}\hat{N}_r \tag{18}$$

5.2. T

$$Eom_4^4 = \frac{(-4t(A(t)-2)(B_t(t)) - 4B(t)(A(t)-1))(F^{'}(T) + F(T)t^2A(t)^2B(t)}{2t^2A(t)^2B(t)} \tag{19}$$

Extrinsic Curvature

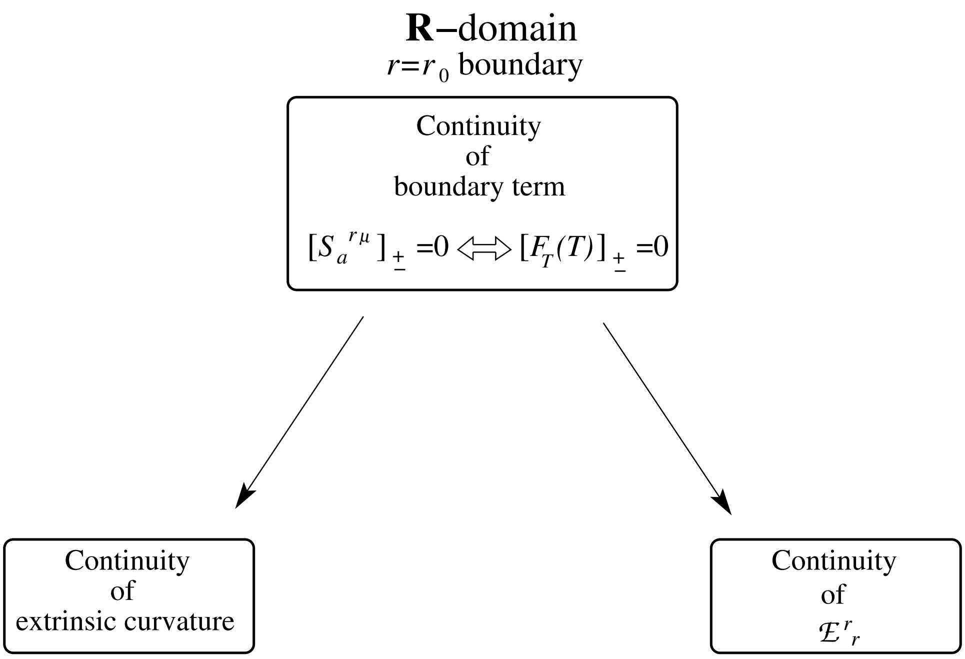
$$K_1^1 = \frac{B_t(t)}{2\sqrt{A(t)}}, K_2^2 = \frac{t}{\sqrt{A(t)}}, K_3^3 = \frac{t\sin^2(\theta)}{\sqrt{A(t)}}, K_4^4 = 0 \tag{20}$$

Boundary Term

$$F'(T){S_a}^{t\nu}\hat{N}_t \tag{21}$$

5.3. Implications

For both cases, if the EOM is continuous, the boundary condition and the extrinsic curvature are. If the extrinsic curvature is continuous the EOM is too. Continuity of the normal boundary term also implies continuity of the equation of motion in the the direction of the component the boundary is dependent on.



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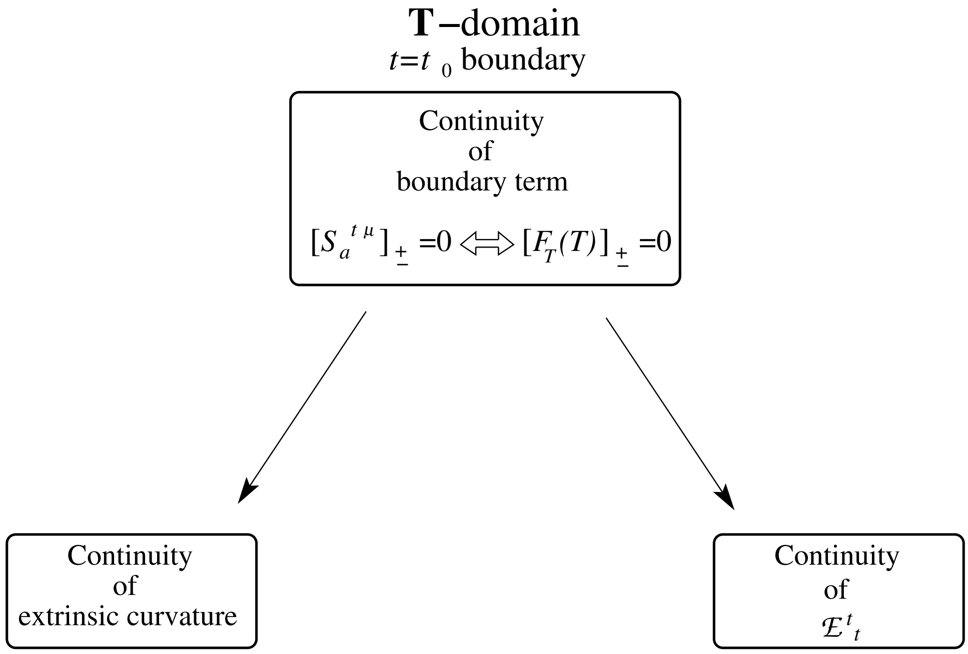

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